Representation Theory of Finite Groups

Back Paper Examination

`December 23, 2024`

Instructions: All questions carry ten marks. Vector spaces and representations are assumed to be finite dimensional over the field of complex numbers. Groups are assumed to be finite.

1. Let R be a ring and M and N be two R-modules. Prove that the tensor product $M \otimes_R N$ is isomorphic to $N \otimes_R M$ as an R-module.
2. Let G be a group of odd order. Let $x \in G$ be a non-identity element. Prove that x^{-1} does not belong to the conjugacy class of x.
3. Let V be an irreducible representation of a finite group G. Determine all G-equivariant linear operators of V.
4. Let G be a non-abelian group of order 55. Determine all one-dimensional representations of G.
5. Let n be a natural number. Prove that the sign representation and the trivial representation are the only one-dimensional representations of the symmetric group on n letters.
6. If V is a representation of a group G such that the only G-invariant linear operators on V are multiplication by scalars, then prove that V is an irreducible representation of G.
7. Let ρ be a representation of G with character χ. Prove that if $\sum_{g\in G} \chi(g) = 0$, then $\sum_{g\in G} \rho(g) = 0$
8. Let H be a subgroup of G. Prove that the permutation representation associated to the left action of G on G/H is induced from the trivial one-dimensional representation W of H.